Tutorial 11

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Week 13

1. Use the Divergence Theorem to find the outward flux of $\mathbf{F}$ across the boundary of the region D.

   $\mathbf{F} = (y - x)\mathbf{i} + (z - y)\mathbf{j} + (y - x)\mathbf{k}$

   $D :$ The cube bounded by the planes $x = \pm 1, y = \pm 1$, and $z = \pm 1$

Suppose S is the surface of the cube D.

$$\text{Flux} = \iint_S \vec{F} \cdot \vec{n} \, d\sigma$$

$$= \iiint_D \nabla \cdot \vec{F} \, dV$$

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(y-x) + \frac{\partial}{\partial y}(z-y) + \frac{\partial}{\partial z}(y-x)$$

$$= -2$$

Therefore

$$\text{Flux} = \iiint_D \nabla \cdot \vec{F} dV$$

$$= \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} -2 \, dz \, dy \, dx$$

$$= -16$$

2. Use the Divergence Theorem to find the outward flux of $\mathbf{F}$ across the boundary of the region D.

   $\mathbf{F} = y\mathbf{i} + xy\mathbf{j} - z\mathbf{k}$

   D : The region inside the solid cylinder $x^2 + y^2 \leq 4$ between the plane $z = 0$ and the paraboloid $z = x^2 + y^2$

$$\text{Flux} = \iint_S \vec{F} \cdot \vec{n}\, d\sigma = \iiint_D \nabla \cdot \vec{F}\, dV.$$

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}(xy) + \frac{\partial}{\partial z}(-z)$$

$$= x - 1$$

Let $x = r\cos\theta$ $\quad y = r\sin\theta$ $\quad 0 \le \theta < 2\pi$ $\quad 0 \le r \le 2$

$0 \le z \le r^2$

$$\text{Flux} = \iiint_D x-1 \, dz\,dy\,dx$$

$$= \int_0^{2\pi}\int_0^2\int_0^{r^2} (r\cos\theta - 1)\, dz\, r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 r^4\cos\theta - r^3 \, dr\,d\theta$$

$$= -8\pi$$

3. Use the Divergence Theorem to find the outward flux of $\mathbf{F}$ across the boundary of the region D.

$\mathbf{F} = \sqrt{x^2 + y^2 + z^2}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$

$D:$ The region $1 \leq x^2 + y^2 + z^2 \leq 2$

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x} x\sqrt{x^2+y^2+z^2} + \frac{\partial}{\partial y} y\sqrt{x^2+y^2+z^2} + \frac{\partial}{\partial z} z\sqrt{x^2+y^2+z^2}$$

$$= \sqrt{x^2+y^2+z^2} + \frac{x^2}{\sqrt{x^2+y^2+z^2}} + \sqrt{x^2+y^2+z^2} + \frac{y^2}{\sqrt{x^2+y^2+z^2}} + \sqrt{x^2+y^2+z^2} + \frac{z^2}{\sqrt{x^2+y^2+z^2}}$$

$$= 4\sqrt{x^2+y^2+z^2}$$

Let $x = \rho \sin\phi \cos\theta \quad y = \rho \sin\phi \sin\theta \quad z = \rho \cos\phi$

$$\text{Flux} = \iint_S \vec{F} \cdot \vec{n}\, d\sigma = \iiint_D \nabla \cdot \vec{F}\, dx\,dy\,dz$$

$$= \int_0^{2\pi} \int_0^{\pi} \int_1^{\sqrt{2}} 4\rho\, (\rho^2 \sin\phi)\, d\rho\, d\phi\, d\theta$$

$$= 12\pi$$

4. If $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is a differentiable vector field, we define the notation $\mathbf{F} \cdot \nabla$ to mean

$$M\frac{\partial}{\partial x} + N\frac{\partial}{\partial y} + P\frac{\partial}{\partial z}.$$

For differentiable vector fields $\mathbf{F}_1$ and $\mathbf{F}_2$, verify the following identities.

a.

$$\nabla \times (\mathbf{F}_1 \times \mathbf{F}_2) = (\mathbf{F}_2 \cdot \nabla)\mathbf{F}_1 - (\mathbf{F}_1 \cdot \nabla)\mathbf{F}_2 + (\nabla \cdot \mathbf{F}_2)\mathbf{F}_1 - (\nabla \cdot \mathbf{F}_1)\mathbf{F}_2$$

b.

$$\nabla(\mathbf{F}_1 \cdot \mathbf{F}_2) = (\mathbf{F}_1 \cdot \nabla)\mathbf{F}_2 + (\mathbf{F}_2 \cdot \nabla)\mathbf{F}_1 + \mathbf{F}_1 \times (\nabla \times \mathbf{F}_2) + \mathbf{F}_2 \times (\nabla \times \mathbf{F}_1)$$

Let $\vec{F}_1 = M_1\vec{i} + N_1\vec{j} + P_1\vec{k}$

$\vec{F}_2 = M_2\vec{i} + N_2\vec{j} + P_2\vec{k}$

a. $\vec{F}_1 \times \vec{F}_2 = \vec{i}(N_1P_2 - P_1N_2) + \vec{j}(M_1P_2 - P_1M_2) + \vec{k}(M_1N_2 - N_1M_2)$

i components of LHS $= \nabla \times (\vec{F}_1 \times \vec{F}_2) = \left[\frac{\partial}{\partial y}(M_1N_2 - N_1M_2) - \frac{\partial}{\partial z}(M_1P_2 - P_1M_2)\right]$

i components of RHS $= \left(M_2\frac{\partial}{\partial x} + N_2\frac{\partial}{\partial y} + P_2\frac{\partial}{\partial z}\right)M_1 - \left(M_1\frac{\partial}{\partial x} + N_1\frac{\partial}{\partial y} + P_1\frac{\partial}{\partial z}\right)M_2$

$+ \left(\frac{\partial}{\partial x}M_2 + \frac{\partial}{\partial y}N_2 + \frac{\partial}{\partial z}P_2\right)M_1 - \left(\frac{\partial}{\partial x}M_1 + \frac{\partial}{\partial y}N_1 + \frac{\partial}{\partial z}P_1\right)M_2$

$= N_2\frac{\partial}{\partial y}M_1 + M_1\frac{\partial}{\partial y}N_2 - N_1\frac{\partial}{\partial y}M_2 - M_2\frac{\partial}{\partial y}N_1 + P_2\frac{\partial}{\partial z}M_1 + M_1\frac{\partial}{\partial z}P_2 - P_1\frac{\partial}{\partial z}M_2 - M_2\frac{\partial}{\partial z}P_1$

$= \frac{\partial}{\partial y}(M_1N_2 - N_1M_2) + \frac{\partial}{\partial z}(M_1P_2 - P_1M_2)$

Same type of results will hold for components $\vec{j}$ and $\vec{k}$

b. Same method as (a).